COMPARISON OF CLASSICAL AND INCURSIVE CONTROL OF LANCHESTER TYPE EQUATIONS WITH VARIABLE COEFFICIENTS

Tomaz Savsek
HQ Military Schools, Ministry of Defense
Republic of Slovenia

Marjan Vezjak
Faculty of Electrical and Computer Engineering
University of Ljubljana, Slovenia

ABSTRACT

The focus of our research and the real purpose of this paper is to compare ecological and military models. We can conclude that both, ecological and military systems are rather complex and that numerous methods, algorithms and experiences of the environment can be used in ecological as well as in military systems. This field offers many opportunities.

The behavior of the ecological and military systems can be simulated by models where sets of differential (Lanchester) equations are applied. The problem of stabilization of unstable equations systems is always presented. We proposed an extension of the classical modeling of combat via Lanchester equations. The usual coupled set of ordinary differential equations is replaced by a system of differential equations where the principle of incursive control is applied.

The mathematical theory of the stabilization of unstable equations systems and the holistic control of systems by incursion is being developed: this new incursive theory confirms the specific examples given in this paper in a more general framework.

AUTHORS BIOGRAPHY

Toma_ SAV_EK was born in Novo mesto, Slovenia, May 11, 1964. He earned his B.S. degree in 1989 and the M.Sc. degree in 1992 in Electrical Science from the University of Ljubljana, Faculty for Electrical and Computer Engineering. Since 1989 he has worked as a researcher in the area of computer vision and pattern recognition on the Faculty for Electrical and Computer Science, Ljubljana. Since 1992 he works as a Professor for Computer Science in the Ministry of Defense, HQ Military Schools. He is in charge of a project which deals with the development and establishing a tactical simulation system for the Slovene Army.

Marjan VEZJAK was born in Maribor, Slovenia, September 2, 1950. He earned his B.S. degree in 1976, M.Sc. degree in 1985 and Ph.D. degree in Electrical Science from the University of Ljubljana, Faculty for Electrical and Computer Engineering. He works as a Research Assistant and Assistant Professor in the Faculty for Electrical and Computer Engineering in Ljubljana. His main research interests are System theory, Signal Processing, Ecological Modeling & Simulation and Sustainable Development.

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1 INTRODUCTION

In the face of global cuts in defense budgets, simulation based training has been recognized internationally as a solution to retain competency whilst reducing training costs. New tasks for armed forces, new scenarios, multinational structures, restrictions of financial and personnel resources as well as those resulting from the protection of the natural environment require new concepts and solutions in the area of military training, exercises and planning. The resulting loss of 'reality' in conventional live exercises due to restrictions in the availability of supporting military personnel and the limitation to reduced training grounds must be compensated as much as possible by modern simulation technology. The second reason for greater use of tactical simulation systems is ecology. This reason is more and more present in some countries which are confronted with small areas reserved for military activities and overcrowding.

Modeling, the process of deriving a model of the real system given by some information of the real system, is not a trivial task. The main reason for this is that through the complexity of the real system it is impossible to comprehend and represent a real system as a whole or even to get complete information of it. Even though model building is much easier for artificial systems or systems to be designed, this is even true here because uncertainties in the physical realization always exist. Therefore, a real system should be conceived as nothing else but an infinite source of observable data from which in a finite time we are able to retrieve just a finite portion. Battle simulations and other military related spheres used in training and planning are very useful areas of research where system approach to solving complex problems with various computer simulation tools can be used.

For the sake of this paper, a system is a battle simulation that makes possible to train commanders and their staff, to examine their knowledge and to try out new tactical principles. A simulation model supports one or two-sided exercises at the operational level. The computer assistance for an exercise can be phase, (b) execution (gaming) phase and (c) analysis phase. Command and control can be automated to a high degree by using the Command, Control, Communications and Information (C^3I) features and Artificial Intelligence (AI) provided by the model. Support weapon systems, like artillery units, are assigned to combat units, so that fire and movements are completely controlled by the model. Positioning and movements are conducted by using maps with the terrain features such as visibility, vegetation covering, accessibility, etc. The number of casualties is calculated by means of Lanchester coefficients and kill probabilities.

2 LANCHESTER QUADRATIC MODEL

Although combat between two military forces is an extremely complex process, since the time of Lanchester analysts have employed simplified models to obtain insights into the dynamics of combat. Complex system models that are possible today because of modern computer technology represent the combat process with a large number of physical and tactical variables in order to describe weapon system performance in a realistic combat environment [Lanchester 1914, Taylor 1971].

The simplest model for the number of survivors in a battle over time is given by the following equations, known as "Lanchester equations" (or Lanchester quadratic model) in the context of wargaming:

$$dX(t)/dt = -\,a_r Y(t)\,,$$
$$dY(t)/dt = -\,a_b X(t)\,, \qquad (2.1)$$

where $X(t)$, $Y(t)$ denote the number of "blue" and "red" survivors at time t, and a_b, a_r the attrition coefficients of "blue" and "red" respectively. The system of ordinary differential equations (2.1) is restricted to the first quadrant, i.e. $X(t)\geq 0$ and $Y(t)\geq 0$. The system (2.1) has the following integral:

i.e. the well-known Lanchester square-law, which leads immediately to the parity condition

$$a_b X^2(0) = a_r Y^2(0) .$$

Numerous generalizations of (2.1) have been proposed, such as heterogeneity of forces, time dependent attrition and non-linearity. The main shortcomings of description (2.1) and related models are their rather unrealistic simplicity and, maybe, their completely deterministic approach [Amacher 1986].

On the other hand, the quadratic model shows how a much more effective job can be made by dividing the enemy and concentrating on the fractions in turn. The quadratic model, according to Lanchester, represents modern warfare. He gives instances drawn from famous battles in the past where victory was obtained because the general was able to divide and concentrate. In another article Lanchester claims that at Trafalgar, Nelson knowingly or unknowingly, employed this principle [Taylor, J.G. & S.H. Pary 1975, Amacher 1986].

3 LANCHESTER-TYPE EQUATIONS WITH VARIABLE COEFFICIENTS

Previous papers sketched some work on variable coefficients Lanchester-type equations and developed a solution for combat between two homogeneous forces in the special case of a constant ratio of the attrition-rate coefficients. This paper develops solutions to extensions of F. W. Lanchester's classical equations in which the lethality (or effectiveness) of fire (as expressed by the Lanchester attrition-rate coefficients) may change over time [Taylor 1971].

This paper considers only the purely formal, mathematical aspects of the Lanchester theory of combat. It describes solutions to extensions of F.W. Lanchester's classical equations for combat between two homogeneous forces. In these extensions the lethality of the fire (as expressed by the Lanchester attrition-rate coefficients) depends on time. When the dependence is arbitrary, the solution is an infinite series of recursively related integrals: in special cases, more convenient representations are available. Solutions are obtained in the following case: the lethality of each side's fire is linear with time, but initially only one side's fire is zero. This case models the constant-speed approach between forces whose weapons have different maximum effective ranges [Sav_ek 1994, Taylor 1971].
Bonder pioneered in the study of range capabilities and mobility considerations for weapon system in combat described by

His results illustrate how his type of analysis may be extended to weapon systems with (a) different range dependencies of lethality of each side's fire (but the same maximal effective range) and (b) linear attrition-rate coefficients but different effective ranges [Taylor 1971].

We consider a constant-speed (v) attack of a defended position with the combat described by:

$$dX/dt = -\alpha(r)Y = -\alpha_0(1 - r/R_\alpha)^m Y ,$$

$$dY/dt = -\beta(r)X = -\beta_0(1 - r/R_\beta)^n X ,$$

where R_α and R_β are maximal effective ranges of the Y and X -force weapon systems, i.e. $\alpha(r)=0$ for $r>R_\alpha$. Range is related to time by $r(t)=R_0-vt$, where R_0 is the opening range of battle, i.e. maximal effective range. Exponents m and n are referred to as the power attrition-rate coefficients.

Taylor studied the dependence of the attrition-rate coefficient $a_r = \alpha(r)$ on the exponent m and n for changeable (different) effective range of the weapon system and constant kill capability at zero range. The exponent m includes nonlinearity of time dependent attrition-rate coefficient $\alpha(r)$ into the model. Linear dependency of the coefficient and time is kept only when $m=1$. The actual value of m is determined in regard to the purpose of the model. The higher m is, the lower the attrition-rate coefficient $a(r)$ is. If $m=0$, then $a(r)=\alpha_0=const.$

4 HOLISTIC CONTROL BY INCURSION

All real systems, which can be reliably modeled by a finite set of or ordinary differential equations could also be modeled by a suitable chosen system of Lotka-Volterra equations which can be derived by transformation of the original description with help of the so-called Structure Design Principle [Peschel 1994]. The focus of our research and the real purpose of this paper is to compare ecological and military models. Numerous methods, algorithms and experiences about the environment can be used in ecological as well as in military system. This field offers many

The behavior of the ecological and military systems can be simulated by models where sets of differential (Lanchester) equations are applied. The problem of stabilization of unstable equations systems is always present. We proposed an extension of the classical modeling of combat via Lanchester equations. The usual coupled set of ordinary differential equations is replaced by a system of differential equations where the principle of incursive control is applied.

The holistic control problem is based on the following basic model considering the two recursive linear processes:

$$x(n+1) = \boldsymbol{W}\ x(n), \qquad (4.1)$$
$$y(n+1) = \boldsymbol{M}\ y(n), \qquad (4.2)$$

where $n = 0,1,2,3,....$ with $x(0)$ and $y(0)$ as the initial conditions, and where $x(n)$ and $y(n)$ are vectors and $\boldsymbol{W}, \boldsymbol{M}$ are matrixes. The equation (4.1) represents the natural or uncontrolled system to be controlled whereas the equation (4.2) is the wanted or controlled system. The incursive control of the system is considered in writing the equation (4.2) in this way:

$$y(n+1) = \mathbf{W}\ [y(n) + \mathbf{Q}\ y(n+1)], \qquad (4.3)$$

where $\boldsymbol{Q}$ is a control matrix.

Let us consider the non-linear model given by the discretized Lotka-Volterra equations:

$$X(t+dt) = X(t) + dt[aX(t) - bX(t)Y(t)], \qquad (4.4)$$
$$Y(t+dt) = Y(t) + dt[-cY(t) + dX(t)Y(t)] \qquad (4.5)$$

where t is a discrete time by steps dt, and a, b, c , d are the parameters.

It is necessary to bear in mint that these equations are related to the Volterra predator-prey ecological system and to the Lotka auto-catalytic chemical reactions system. Volterra-Lotka describes interactions between preys and density X with predators and density Y by means of differential equations (Gold, 1977; Metzler, 1987). The solutions of the original differential equations system are given by periodic oscillations (orbital stability). In the linearised case, the variable Y has a phase delay of $\Pi/2$ on the variable X. The numerical simulations of equations (4.4) and (4.5) are unstable.
There are different incursive algorithms which control the oscillations of these discretized equations. The iterative values of $X(t+dt)$ of the first equation can be propagated to the second equation in an incursive way, as proposed by Dubois:

$$X(t+dt)=X(t)+dt[aX(t)-bX(t)Y(t)], \qquad (4.6)$$
$$Y(t+dt)=Y(t)+dt[-cY(t)+dX(t+dt)Y(t)], \qquad (4.7$$

where we compute the value $Y(t+dt)$ in function of the value of $X(t+dt)$ at the same time step, instead of the value at its preceding step as it is classically done with a recursive parallel way. The name of "incursion" means "inclusive recursion" and corresponds to temporal asynchronous iterations. Notice that for small values of the time step dt, which correspond to the continuous case, we obtain the same results as the original Lotka-Volterra equations. But, for non-zero values of the time step, the simulation of this model gives very surprising solutions. The numerical solutions show stabilized oscillations by incursion [Dubois 1994].

We should notice that for a lot of values of the parameters, the incursive solutions give stable oscillations, meanwhile the recursive ones give numerically unstable solutions.

5 COMPUTER SIMULATION OF THE CONSTANT-SPEED ATTACK

For the purpose of this paper we simulated the constant-speed attack by the simulations tools MATLAB® and SIMULINK® (registered trademarks of The MathWorks, Inc.). MATLAB® is a technical computing environment for high-performance numeric computation and visualization. SIMULINK® is a program for simulating dynamic systems. As a extension to MATLAB®, SIMULINK® adds many features specific to dynamic systems while retaining all of MATLAB®'s general purpose functionality.

At the simulation we used the system of differential (Lanchester) equations:

$$dX/dt = -\alpha(r)Y = -\alpha_0(1 - r/R_\alpha)^m Y,$$
$$dY/dt = -\beta(r)X = -\beta_0(1 - r/R_\beta)^n X,$$

where parameters are defined as follows:

- The maximal effective range of weapon system: $R_0=R_\alpha=R_\beta=2000$ meters.
- Initial attrition-rate coefficient $a(r=0) = \alpha_0 = 0.6$ X - casualties / (unit time × Y - unit) that denotes the Y - force weapon-system

- Initial attrition-rate coefficient $b(r=0) = \beta_0 = 0.6$ Y-casualties / (minutes × X-unit) that denotes the X - force weapon-system kill rate at zero force separation (range).
- The constant-speed: $v=5$ m/h (8.045 km/h).
- The initial number of the X - force weapon-system: $X_0=10$.
- The initial number of the Y - force weapon-system: $Y_0=30$.
- The exponents: $m=2$, $n=1$.

An approach to solving of the system of differential equations can be presented in three steps:

- step 1: equations must be rearranged so that the highest differentials are on the left side, all the others element are on the right side of equations (e.g. in state space),
- step 2: we take as many integrators as necessary to get all variables that appear in the system,
- step 3: we construct a complete simulation schematic as shown on Fig.1.

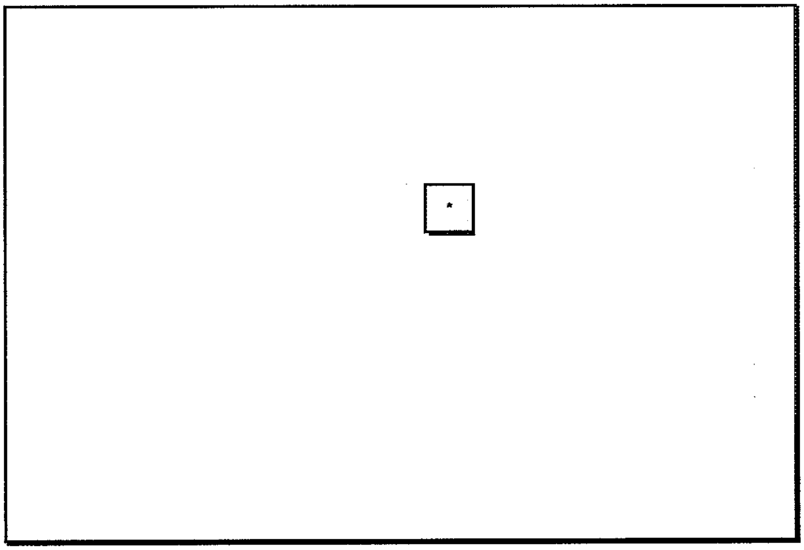

Fig. 1: Complete simulation schematic for the constant-speed attack

Fig.2 shows the force-level trajectories of Y and X. Trajectories are the results of the simulation by using the classical control. Fig.3 shows the force-level trajectories of Y and X, where the incursive control is applied. The numerical results for classical control (CC) and incursive control (IC) can be seen in Table 1.

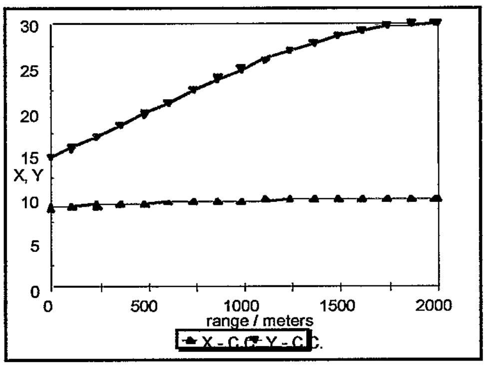

Fig.2: Force-level trajectories of the X and Y (classical control)

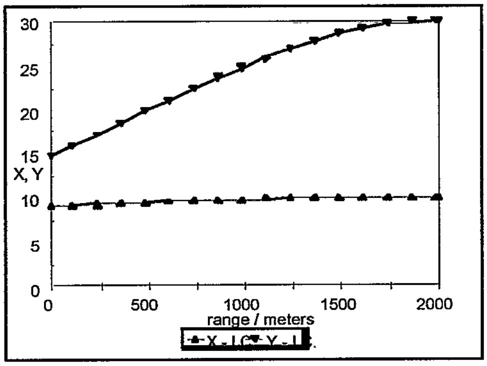

Fig.3: Force-level trajectories of the X and Y (incursive control)

We can realized from Fig.2, Fig.3 and Table 1 that there are only small differences between classical and incursive control. This fact is due to a very short time of simulation and a rather simple (not complex) system. In a case of greater complexity (more units involved - more differential equations) or longer time of combat

Range [m]	X - C.C	Y - C.C.	X - I.C.	Y - I.C.
2000	10.000	30.000	10.000	30.000
1984	10.000	29.999	10.000	29.999
1859	10.000	29.890	10.000	29.890
1734	9.997	29.608	9.997	29.608
1609	9.989	29.159	9.989	29.159
1484	9.975	28.550	9.975	28.552
1359	9.953	27.794	9.953	27.797
1234	9.922	26.903	9.922	26.909
1109	9.880	25.894	9.880	25.904
984	9.827	24.785	9.828	24.800
859	9.763	23.595	9.764	23.616
734	9.688	22.342	9.689	22.372
609	9.602	21.047	9.604	21.086
484	9.506	19.728	9.509	19.777
359	9.401	18.403	9.405	18.463
234	9.289	17.087	9.294	17.157
109	9.171	15.795	9.176	15.875
0	9.064	14.692	9.070	14.779

Table 1: General changes across the range for trajectories of X and Y

We took longer time of simulation and smaller coefficients, so the transitional phenomena of the system was longer. In that case the differences are more significant. Fig.4 and Table 2 represent the results of simulation with (16 times) longer time. If we look at the force-level trajectories of the Y more precisely, we can realize that the trajectory of the incursive control is a little higher than the trajectory of the classical control. Why?! The answer is in the principle of the incursive control. At the incursive control we compute the value $Y(t+dt)$ in function of the value of $X(t+dt)$ at the same time step, instead of the value at its preceding step as it is classically done with a recursive parallel way. The trajectory of the X - forces is almost the same in classical and incursive control. That is because we compute the value $X(t+dt)$ in function of the value $Y(t)$.

Practically, that means that the Y - forces use not only the data from the past but also the data and information form the present. They can easily predict the behavior of the X - forces. So they have less casualties as in classical way.

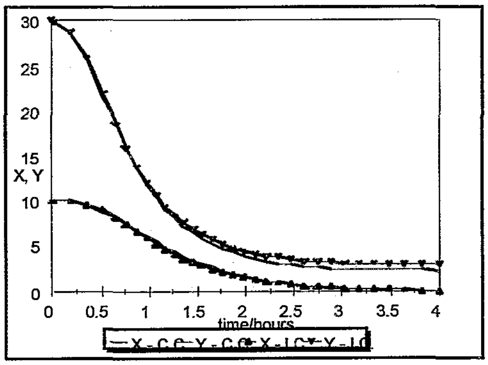

Fig.4: Force-level trajectories of the X and Y (CC and IC)

Time [hours]	X - C.C	Y - C.C.	X - I.C.	Y - I.C
0.00	10.000	30.000	10.000	30.000
0.03	10.000	29.965	10.000	29.965
0.20	9.928	28.635	9.929	28.644
0.36	9.591	25.701	9.601	25.796
0.53	8.857	21.641	8.908	21.972
0.66	8.097	18.478	8.028	18.411
0.77	7.367	15.945	7.265	15.892
0.88	6.604	13.652	6.502	13.711
0.98	5.925	11.842	5.817	11.975
1.08	5.310	10.358	5.198	10.556
1.17	4.746	9.113	4.633	9.370
1.27	4.228	8.056	4.115	8.364
1.36	3.750	7.152	3.640	7.506
1.46	3.311	6.376	3.203	6.770
1.56	2.906	5.707	2.803	6.139
1.66	2.534	5.131	2.437	5.597
1.76	2.193	4.634	2.103	5.133
1.87	1.881	4.206	1.801	4.736
1.98	1.596	3.839	1.529	4.399
2.10	1.339	3.525	1.287	4.115
2.22	1.111	3.262	1.074	3.879
2.35	0.910	3.043	0.889	3.684
2.48	0.734	2.862	0.729	3.524
2.61	0.583	2.714	0.592	3.393
2.74	0.454	2.594	0.476	3.288
2.88	0.345	2.498	0.379	3.203
3.03	0.256	2.423	0.298	3.136
3.17	0.184	2.365	0.232	3.084
3.33	0.128	2.322	0.179	3.044
3.48	0.085	2.291	0.137	3.013
3.65	0.053	2.269	0.104	2.989
3.82	0.031	2.254	0.078	2.972
4.00	0.017	2.245	0.058	2.959

Table 2: General changes during time for trajectories of X and Y

6 DISCUSSION

The world-wide accepted political and humanistic standpoint stresses that to start the war means to commit a crime against the mankind; additionally, the losses of those who start the war should be higher than the profits they could have from it. Apart from this threat, the would-be initiators of wars should be given rational, substantially supported reasons showing clearly that the war serves no purpose anymore, if ever it has. The efficient psychological way of deterrence could be to predict the war outcome on the basis of the initial power ratio between the warring parties. Thus, negotiations could replace fighting. The war raging in the former Yugoslavia could be a proof that either such an analysis was not done or was not taken into consideration. The consequences are well known and tragic [Vezjak & Sav_ek, 1994].

Our goal was not to increase combat efficiency, which is usually the purpose of such research, but to give more credibility to the results which could prove that war is anachronism that should be replaced by negotiations. Violation of peace has a similar effect to that of ecological system balance destruction. The fact that the same simulation tool could be applied to both ecological and military models has connected them at the methodological level. Moreover, their similarity is based on many intertwined factors as well as on their sensitivity, instability, etc.

When modeling the mathematical models of real systems we are frequently confronted with the complicate complexity of such systems. Yet a great majority of the models can be transformed to one of the Volterra-Lotka models. As these models very frequently occur in real systems it is very interesting to make a comparison of some tools being available to solve such equation systems [Peschel, 1994]

System approach to combat is a basis on which tactical or strategic simulation systems are being developed. The process of development and preparation of the tactical simulation system that would include the strategic planning function is an extremely demanding process, which requires the knowledge of experts from numerous areas.

The efficient application of such a system depends on the data structure containing the description of the military organizations. On one hand, the data structure should allow the relevant description of the engaged military units. On the other hand, it should provide for the efficient extraction of the data from the database containing the solved examples.

Provided the relevant database is available, when gaming the tactical assignments, which are carried out by the military experts, we create an expert base, on which the decision-making system is based. The military organization is a typical hierarchical structure. Nevertheless, two facts should be taken into consideration: (i) military structures are relatively complex and nonhomogeneous organizations and (ii) many times, especially when the description of the enemy forces is involved, data is lacking or we deal with the information that is either not correct or even intentionally false. In such a case, the description of the military organization for instance takes the form of a tree relational structure, namely, the form of a fuzzy tree. Thus, the problem of comparing armed forces of the warring parties is transformed into the problem of comparing fuzzy tree structures.

The measurement of the covering and the distance between the fuzzy trees, which represent the military structures, will allow the comparison of the current situation on the battlefield with the previously simulated situations. Thus, a new tool, which will be used in the decision-making system, will be created [Sav_ek & Vezjak, 1995].

The gaming of diverse situations, based on the real data, will enable strategic planning as well as the prediction of the likely combat outcomes. The capability to plan actions and predict the outcome of a possible combat will allow the commanders to act reasonably in order to avoid the unnecessary casualties and damage.

7 CONCLUSION

This paper suggested the convenience of the system approach to the solution of complex Volterra-Lotka problems. The sample models were simulated and gave the expected results. All this represents a solid base for further work on this rather unknown and unexplored field of work.

More real simulation of the considered complex problem solving method would be presented by the actual work on data taken from a real complex problem. To estimate the correctness of presumptions and models, and to complete the models, the feed back information is urgently necessary in the system approach, as well as the comparison of models with actual circumstances in the "military" and "non-military" system.

The focus of our research and the real purpose of this paper is to compare ecological and military models. This field offers many opportunities. The usual analysis by means of Volterra-Lotka and Lanchester type equations with variable coefficients was methodologically compared by MATLAB® simulation tools. The mathematical theory of the stabilization of unstable equations systems and the holistic control of systems by incursion is being developed. We have also pointed to the problem of comparing armed forces of the warring parties which can be transformed into the problem of comparing fuzzy tree structures.

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